



# Separation theorems for the zeros of certain hypergeometric polynomials

K. Driver<sup>a,1</sup>, K. Jordaan<sup>b,\*,2</sup>

<sup>a</sup>John Knopfmacher Centre for Applicable Analysis and Number Theory, School of Mathematics, University of the Witwatersrand, Johannesburg, South Africa

<sup>b</sup>Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria, 0002, South Africa

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## Abstract

We study interlacing properties of the zeros of two contiguous  ${}_2F_1$  hypergeometric polynomials. We use the connection between hypergeometric  ${}_2F_1$  and Jacobi polynomials, as well as a monotonicity property of zeros of orthogonal polynomials due to Markoff, to prove that the zeros of contiguous hypergeometric polynomials separate each other. We also discuss interlacing results for the zeros of  ${}_2F_1$  and those of the polynomial obtained by shifting one of the parameters of  ${}_2F_1$  by  $\pm t$  where  $0 < t < 1$ .

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## 1. Introduction

The general hypergeometric function  ${}_pF_q$  with  $p$  numerator and  $q$  denominator parameters is defined by

$${}_pF_q \left( \begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix}; x \right) = 1 + \sum_{k=1}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_p)_k x^k}{(b_1)_k (b_2)_k \dots (b_q)_k k!}, \quad |x| < 1,$$

where

$$(\alpha)_k = \alpha(\alpha+1) \dots (\alpha+k-1), \quad k \geq 1, \quad k \in \mathbb{N}$$

is Pochhammer's symbol.

If one of the numerator parameters is equal to a negative integer, say  $a_1 = -n$ ,  $n \in \mathbb{N}$ , then the series terminates and is a polynomial of degree  $n$  in  $x$ . A natural, and often very important, question that arises in the study of polynomials is

\* Corresponding author.

E-mail addresses: [kad@science.uct.ac.za](mailto:kad@science.uct.ac.za) (K. Driver), [kjordaan@up.ac.za](mailto:kjordaan@up.ac.za) (K. Jordaan).

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the nature and location of their zeros. The connection between  ${}_2F_1$  hypergeometric polynomials and orthogonal polynomials, particularly the Jacobi polynomials, leads to fairly complete information about the zeros of  ${}_2F_1$  polynomials. However, there is no general link between  ${}_3F_2$  polynomials and classical orthogonal polynomials and the investigation of the location of the zeros of  ${}_3F_2$  polynomials is far more challenging.

Some classes of  ${}_3F_2$  polynomials can be expressed as products of  ${}_2F_1$  polynomials (cf. [5, p. 498]) and the location of their zeros in various intervals on the real line can be deduced (cf. [3,4]). When generating numerical data for the zeros of these classes of  ${}_3F_2$  polynomials, it appeared that in some cases the real, simple zeros of the  ${}_2F_1$  polynomials which are factors of the  ${}_3F_2$  polynomials are interlacing. This observation posed the question of when and how interlacing of the zeros of  ${}_2F_1$  polynomials takes place.

In this paper, we examine the question of the interlacing properties of the zeros of so-called contiguous  ${}_2F_1$  hypergeometric polynomials. The six functions

$${}_2F_1\left(\begin{matrix} a \pm 1, b \\ c \end{matrix}; x\right), \quad {}_2F_1\left(\begin{matrix} a, b \pm 1 \\ c \end{matrix}; x\right) \quad \text{and} \quad {}_2F_1\left(\begin{matrix} a, b \\ c \pm 1 \end{matrix}; x\right)$$

are contiguous to

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right)$$

(cf. [6]) and there are identities that link

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right)$$

with any pair of its contiguous functions via a linear relation in  $x$  (cf. [6, p. 72]). We shall use the known interlacing properties of Jacobi polynomials of successive degrees and also the monotonicity result of Markoff (cf. [7, p. 116, Theorem 6.12.2]) to prove our results.

The Jacobi polynomials  $\mathcal{P}_n^{(\alpha, \beta)}(x)$  and  ${}_2F_1$  polynomials are linked by (cf. [1, p. 295])

$$\mathcal{P}_n^{(\alpha, \beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1\left(\begin{matrix} -n, 1+\alpha+\beta+n \\ 1+\alpha \end{matrix}; \frac{1-x}{2}\right).$$

For  $\alpha, \beta > -1$ , the sequence  $\{\mathcal{P}_n^{(\alpha, \beta)}(x)\}_{n=1}^{\infty}$  of Jacobi polynomials is orthogonal on  $(-1, 1)$  with respect to the weight function  $(1-x)^\alpha(1+x)^\beta$  (cf. [1, p. 299, Theorem 6.4.3]). Correspondingly,

$${}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}; x\right)$$

is orthogonal to all polynomials of lower degree with respect to the (varying with  $n$ ) weight function  $x^{c-1}(1-x)^{b-c-n}$  on  $(0, 1)$  for  $c > 0$  and  $b > c + n - 1$ ; on  $(1, \infty)$  for  $b < 1 - n$  and  $c < b + 1 - n$  with respect to weight function  $x^{c-1}(x-1)^{b-c-n}$ ; and with respect to  $(-x)^{c-1}(1-x)^{b-c-n}$  on  $(-\infty, 0)$  for  $b < 1 - n$  and  $c > 0$  (cf. [2]).

We shall assume throughout our discussion that  $b, c \in \mathbb{R}$  and denote the functions contiguous to

$$F_n(x) = {}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}; x\right)$$

by  $F_n(b+; x)$  etc. where

$$F_n(b+; x) = {}_2F_1\left(\begin{matrix} -n, b+1 \\ c \end{matrix}; x\right).$$

## 2. Separation theorems for the zeros of contiguous hypergeometric polynomials

Our first result is just a direct translation of the known interlacing properties of the zeros of the Jacobi polynomials  $\mathcal{P}_n^{(\alpha, \beta)}(x)$  and  $\mathcal{P}_{n+1}^{(\alpha, \beta)}(x)$  for  $\alpha, \beta > -1$  (cf. [1, p. 253, Theorem 5.4.2]) to the corresponding  ${}_2F_1$  hypergeometric polynomials.

**Theorem 2.1.** Let  $n \in \mathbb{N}$ ,  $b, c \in \mathbb{R}$  and  $c \notin \mathbb{Z}^-$  and let

$$F_n(x) = {}_2F_1 \left( \begin{matrix} -n, b \\ c \end{matrix}; x \right).$$

Then

- (i) for  $c > 0$  and  $b > c + n - 1$ , the zeros of  $F_n(x)$  and  $F_{n+1}(b+; x)$  are interlacing on  $(0, 1)$ ,
- (ii) for  $b < 1 - n$  and  $c < b + 1 - n$ , the zeros of  $F_n(x)$  and

$${}_2F_1 \left( \begin{matrix} -n-1, b-1 \\ c-2 \end{matrix}; x \right)$$

are interlacing on  $(1, \infty)$ ,

- (iii) for  $b < 1 - n$  and  $c > 0$ , the zeros of  $F_n(x)$  and  $F_{n+1}(b-; x)$  are interlacing on  $(-\infty, 0)$ .

**Proof.**

- (i) One of the representations of a Jacobi polynomial in terms of a hypergeometric polynomial is given by (cf. [6, p. 254, eqn. 3])

$$\mathcal{P}_n^{(\alpha, \beta)}(x) = (-1)^n \frac{(1 + \beta)_n}{n!} {}_2F_1 \left( \begin{matrix} -n, b \\ c \end{matrix}; \frac{1+x}{2} \right),$$

where  $b = 1 + \alpha + \beta + n$  and  $c = \beta + 1$ . The conditions  $\alpha > -1$  and  $\beta > -1$  are equivalent to  $c > 0$  and  $b > c + n - 1$  and the interval  $[-1, 1]$  is transformed into the interval  $[0, 1]$  under the linear mapping  $z = (1+x)/2$ . In addition, the transformation is order preserving, since  $z_0 < z_1$  implies that  $x_0 < x_1$ . Now, with  $z = (1+x)/2$ , we have

$$\begin{aligned} \mathcal{P}_{n+1}^{(\alpha, \beta)}(x) &= (-1)^{n+1} \frac{(1 + \beta)_{n+1}}{(n+1)!} {}_2F_1 \left( \begin{matrix} -n-1, 1 + \alpha + \beta + n + 1 \\ \beta + 1 \end{matrix}; z \right) \\ &= (-1)^{n+1} \frac{(1 + \beta)_{n+1}}{(n+1)!} F_{n+1}(b+; z). \end{aligned}$$

Since the zeros of  $\mathcal{P}_n^{(\alpha, \beta)}$  and  $\mathcal{P}_{n+1}^{(\alpha, \beta)}$  are interlacing on  $(-1, 1)$ , it then follows immediately that the zeros of  $F_n(x)$  and  $F_{n+1}(b+; x)$  are interlacing on  $(0, 1)$ .

- (ii) Another representation of a Jacobi polynomial as a hypergeometric polynomial is given by (cf. [6, p. 255, eqn. 7])

$$\mathcal{P}_n^{(\alpha, \beta)}(x) = \frac{(1 + \alpha + \beta)_{2n}}{n!(1 + \alpha + \beta)_n} \left( \frac{x-1}{2} \right)^n {}_2F_1 \left( \begin{matrix} -n, b \\ c \end{matrix}; \frac{2}{1-x} \right),$$

where  $b = -\alpha - n$  and  $c = -\alpha - \beta - 2n$ . Under the fractional transformation  $z = 2/(1-x)$ , the interval  $-1 < x < 1$  is mapped to  $1 < z < \infty$ . Also  $\alpha > -1$  and  $\beta > -1$  corresponds to  $b < 1 - n$  and  $c < b + 1 - n$ . The transformation is order preserving since  $x_0 < x_1$  implies that  $z_0 < z_1$  since  $x_0, x_1 \in (-1, 1)$ . Now, with  $z = 2/(1-x)$ , we have

$$\begin{aligned} \mathcal{P}_{n+1}^{(\alpha, \beta)}(x) &= \frac{(n+1)!x^{n+1}}{(c)_{n+1}} {}_2F_1 \left( \begin{matrix} -n-1, -n-\alpha-1 \\ -2n-\alpha-\beta-2 \end{matrix}; z \right) \\ &= \frac{(n+1)!x^{n+1}}{(c)_{n+1}} {}_2F_1 \left( \begin{matrix} -n-1, b-1 \\ c-2 \end{matrix}; z \right) \end{aligned}$$

which proves the result.

- (iii) The representation (cf. [6, p. 254, eqn. 2])

$$\mathcal{P}_n^{(\alpha, \beta)}(x) = \frac{(1 + \alpha)_n}{n!} \left( \frac{x+1}{2} \right)^n F_n \left( \frac{x-1}{x+1} \right),$$

where  $b = -\beta - n$  and  $c = 1 + \alpha$  yields the stated result, since  $\alpha, \beta > -1$  will correspond to  $b < 1 - n$  and  $c > 0$  and the interval  $-1 < x < 1$  is mapped to the negative real line under the transformation  $z = (x - 1)/(x + 1)$ . Since  $x + 1 > 0$ , it can easily be checked that the transformation is order preserving. Now, with  $z = (x - 1)/(x + 1)$ , we have

$$\begin{aligned}\mathcal{P}_{n+1}^{(\alpha, \beta)}(x) &= \frac{(1 + \alpha)_{n+1}}{(n + 1)!} \left( \frac{x + 1}{2} \right)^{n+1} {}_2F_1 \left( \begin{matrix} -n - 1, -\beta - n - 1 \\ \alpha + 1 \end{matrix}; z \right) \\ &= \frac{(1 + \alpha)_{n+1}}{(n + 1)!} \left( \frac{x + 1}{2} \right)^{n+1} F_{n+1}(b-; z)\end{aligned}$$

and the result follows.  $\square$

In what follows, we will state and prove the results for the interval  $(0, 1)$  but note that similar results hold for the interval  $(-\infty, 0)$ , with the appropriate corresponding restrictions on  $b$  and  $c$  as stated in Theorem 2.1. We note that throughout our discussions, we take the positive real value of any power, for example the positive real value of  $x^{c-1}(1-x)^{b-c-n}$ ,  $x \in (0, 1)$ ,  $b, c \in \mathbb{R}$ ,  $n \in \mathbb{N}$ .

**Theorem 2.2.** Let  $c > 0$  and  $b > c + n - 1$  and let

$$F_n(x) = {}_2F_1 \left( \begin{matrix} -n, b \\ c \end{matrix}; x \right).$$

Suppose that  $0 < x_1 < x_2 < \dots < x_n < 1$  are the zeros of  $F_n(x)$  and  $0 < X_1 < X_2 < \dots < X_n < 1$  are the zeros of  $F_n(b+; x)$ . Then

$$X_k < x_k \quad \text{for each } k = 1, 2, \dots, n \quad (1)$$

and

$$x_k < X_{k+1} \quad \text{for } k = 1, 2, \dots, n - 1. \quad (2)$$

**Proof.** For  $c > 0$  and  $b > c + n - 1$  the polynomial  $F_n(x)$  is orthogonal on  $(0, 1)$  with respect to the positive, continuous weight function

$$w_1(x) = x^{c-1}(1-x)^{b-c-n} \quad (3)$$

whereas the orthogonality of  $F_n(b+; x)$  is with respect to

$$w_2(x) = x^{c-1}(1-x)^{b-c-n+1}. \quad (4)$$

Therefore, the ratio

$$\frac{w_1(x)}{w_2(x)} = \frac{1}{1-x}$$

is an increasing function of  $x$  on  $(0, 1)$ . The monotonicity result of Markoff (cf. [7, p. 116, Theorem 6.12.2]) then shows that  $X_k < x_k$  for each  $k = 1, 2, \dots, n$  which proves (1). Next, consider the contiguous function relation (cf. [6, p. 72, Exercise 23])

$$(c - 1 - b)F_n(x) = (c + n)F_{n+1}(b+; x) + (-n - 1 - b)(1 - x)F_n(b+; x).$$

At each of the zeros  $x_k$ ,  $k = 1, \dots, n$ , of  $F_n(x)$ , we have

$$(c + n)F_{n+1}(b+; x_k) = (b + n + 1)(1 - x_k)F_n(b+; x_k),$$

and it follows that

$$\begin{aligned}(c + n)^2 F_{n+1}(b+; x_k) F_{n+1}(b+; x_{k+1}) \\ = (b + n + 1)^2 (1 - x_k)(1 - x_{k+1}) F_n(b+; x_k) F_n(b+; x_{k+1}).\end{aligned} \quad (5)$$

By Theorem 2.1, we know that the zeros  $\{x_k\}_{k=1}^n$  of  $F_n(x)$  interlace with those of  $F_{n+1}(b+; x)$  and therefore  $F_{n+1}(b+; x)$  has a different sign at successive zeros of  $F_n(x)$ . We may then deduce from (5), since  $(1-x_k)(1-x_{k+1}) > 0$ , that

$$F_n(b+; x_k)F_n(b+; x_{k+1}) < 0 \quad \text{for each } k = 1, 2, \dots, n-1$$

which implies that  $F_n(b+; x)$  has at least one zero of odd order between  $x_k$  and  $x_{k+1}$ . In conjunction with (1), we deduce that

$$x_k < X_{k+1} < x_{k+1} \quad \text{for } k = 1, \dots, n-1$$

and (2) is proved.  $\square$

**Theorem 2.3.** For  $c > 0$  and  $b > c + n - 1$ , the zeros of  $F_n(x)$  and  $F_{n-1}(x)$  separate each other on  $(0, 1)$ .

**Proof.** Evaluating the contiguous function relation (cf. [6, p. 71, Exercise 21(1)])

$$(-n-b)F_n(x) = -nF_{n-1}(x) - bF_n(b+; x)$$

at successive zeros  $x_k$  and  $x_{k+1}$  of  $F_n(x)$ , we obtain

$$n^2 F_{n-1}(x_k)F_{n-1}(x_{k+1}) = b^2 F_n(b+; x_k)F_n(b+; x_{k+1}).$$

By Theorem 2.2, the zeros of  $F_n(x)$  and  $F_n(b+; x)$  separate each other which implies that

$$F_{n-1}(x_k)F_{n-1}(x_{k+1}) < 0 \quad \text{for } k = 1, \dots, n-1$$

and the stated result follows.  $\square$

**Corollary 2.4.** For  $c > 0$  and  $b > c + n$ , the zeros of  $F_n(x)$  and  $F_{n+1}(x)$  separate each other on  $(0, 1)$ .

**Proof.** This follows immediately from Theorem 2.3 by replacing  $n$  with  $n+1$ .  $\square$

**Theorem 2.5.** For  $c > 0$  and  $b > c+n$  the zeros  $0 < x_1 < \dots < x_n < 1$  of  $F_n(x)$  and  $0 < t_1 < t_2 < \dots < t_n < 1$  of  $F_n(b-; x)$  are interlaced as follows:

$$x_k < t_k \quad \text{for } k = 1, \dots, n \quad \text{and} \quad t_k < x_{k+1} \quad \text{for } k = 1, 2, \dots, n-1.$$

**Proof.** The weight function for  $F_n(b-; x)$  on  $(0, 1)$  is  $w_0(x) = x^{c-1}(1-x)^{b-c-n-1}$  for  $c > 0$  and  $b > c+n$ . Hence the ratio of the appropriate weight functions with  $w_1(x)$  as in (3), is

$$\frac{w_0(x)}{w_1(x)} = \frac{1}{1-x}.$$

This is an increasing function of  $x$  on  $(0, 1)$  and we deduce from Markoff's monotonicity result that

$$x_k < t_k \quad \text{for } k = 1, 2, \dots, n. \tag{6}$$

Next, by evaluating the contiguous function relation (cf. [6, p. 71, Exercise 21(7)])

$$(b-c-n)F_n(x) = n(x-1)F_{n-1}(x) - (c-b)F_n(b-; x)$$

at successive zeros  $x_k$  and  $x_{k+1}$ ,  $k \in \{1, 2, \dots, n-1\}$  of  $F_n(x)$ , we have for  $k = 1, 2, \dots, n-1$

$$n^2(x_k-1)(x_{k+1}-1)F_{n-1}(x_k)F_{n-1}(x_{k+1}) = (c-b)^2 F_n(b-; x_k)F_n(b-; x_{k+1}). \tag{7}$$

We know from Theorem 2.2 that the zeros of  $F_n(x)$  and  $F_{n-1}(x)$  are interlacing and since  $(x_k-1)(x_{k+1}-1) > 0$ , the left-hand side of (7) is negative. Hence

$$F_n(b-; x_k)F_n(b-; x_{k+1}) < 0$$

which implies that  $F_n(b-; x)$  has at least one zero of odd order between  $x_k$  and  $x_{k+1}$  for each  $k = 1, \dots, n-1$ . Together with (6), we deduce that

$$x_k < t_k < x_{k+1} \quad \text{for } k = 1, 2, \dots, n-1$$

which completes the proof.  $\square$

**Theorem 2.6.** For  $c > 1$  and  $b > c + n - 1$ , the zeros  $0 < x_1 < \dots < x_n < 1$  of  $F_n(x)$  and  $0 < r_1 < r_2 < \dots < r_n < 1$  of  $F_n(c-; x)$  are interlaced as follows:

$$r_k < x_k \quad \text{for } k = 1, 2, \dots, n \quad \text{and} \quad x_k < r_{k+1} \quad \text{for } k = 1, \dots, n-1.$$

**Proof.** The polynomial  $F_n(c-; x)$  is orthogonal on the interval  $(0, 1)$  with respect to the weight function  $w_3(x) = x^{c-2}(1-x)^{b-c-n+1}$  for  $c > 1$  and  $b > c + n - 2$ . The ratio of the weight functions of  $F_n(c-; x)$  and  $F_n(x)$ , with  $w_1(x)$  as in (3), is

$$\frac{w_1(x)}{w_3(x)} = \frac{x}{1-x}$$

which is an increasing function of  $x$  on  $(0, 1)$ . Hence we know (cf. [7, p. 116, Theorem 6.12.2]) that  $r_k < x_k$  for  $k = 1, \dots, n$ .

To show that  $x_k < r_{k+1}$  for  $k = 1, \dots, n-1$ , we use the contiguous function relation (cf. [6, p. 71, Exercise 21(2)])

$$(-n - c + 1)F_n(x) = -nF_{n-1}(x) - (c - 1)F_n(c-; x)$$

and proceed as in the proof of Theorem 2.2, using the result proved in Theorem 2.3 that the zeros of  $F_n(x)$  and  $F_{n-1}(x)$  interlace.  $\square$

**Theorem 2.7.** For  $c > 0$  and  $b > c + n$  the zeros  $0 < x_1 < \dots < x_n < 1$  of  $F_n(x)$  and  $0 < p_1 < p_2 < \dots < p_n < 1$  of  $F_n(c+; x)$  are interlaced as follows:

$$x_k < p_k \quad \text{for } k = 1, \dots, n \quad \text{and} \quad p_k < x_{k+1} \quad \text{for } k = 1, \dots, n-1.$$

**Proof.** The polynomial  $F_n(c+; x)$  is orthogonal on  $(0, 1)$  with respect to  $w_4(x) = x^c(1-x)^{b-c-n-1}$  for  $c > -1$  and  $b > c + n$ . The ratio of the appropriate weight functions, with  $w_1(x)$  given by (3), is

$$\frac{w_4(x)}{w_1(x)} = \frac{x}{1-x}$$

which is an increasing function of  $x$  on  $(0, 1)$ . Hence we have  $x_k < p_k$  for  $k = 1, \dots, n$ .

Using the contiguous function relation (cf. [6, p. 72, Exercise 21(4)])

$$(1-x)F_n(x) = F_{n+1}(x) - \frac{c-b}{c}x F(c+; x)$$

in place of (6), the proof then parallels that of Theorem 2.5, noting that the zeros of  $F_n(x)$  and  $F_{n+1}(x)$  interlace by Corollary 2.4.  $\square$

**Theorem 2.8.** For  $c > 0$  and  $b > c + n - 1$  the zeros  $0 < x_1 < \dots < x_n < 1$  of  $F_n(x)$  and  $0 < s_1 < s_2 < \dots < s_n < 1$  of  $F_n(b+, c+; x)$  are interlaced as follows:

$$x_k < s_k \quad \text{for } k = 1, \dots, n \quad \text{and} \quad s_k < x_{k+1} \quad \text{for } k = 1, \dots, n-1.$$

**Proof.** We know that  $F_n(b+, c+; x)$  is orthogonal on  $(0, 1)$  with respect to  $w_5(x) = x^c(1-x)^{b-c-n}$  for  $c > -1$  and  $b > c + n - 1$ . Then, the ratio of the weight functions

$$\frac{w_5(x)}{w_1(x)} = x$$

is an increasing function of  $x$  on  $(0, 1)$ , where  $w_1(x)$  is given in (3). Hence we have  $x_k < s_k$  for  $k = 1, \dots, n$ .

Also, using the contiguous function relation (cf. [6, p. 72, Exercise 22])

$$F_n(x) = F_{n+1}(b+; x) + \frac{(b+1+n)}{c} x F_n(b+, c+; x)$$

in place of Eq. (6), the proof then parallels that of Theorem 2.5, noting that  $x_k, x_{k+1} > 0$  and using the result from Theorem 2.1 that the zeros of  $F_n(x)$  and  $F_{n+1}(b+; x)$  separate each other.  $\square$

### 3. An interesting generalisation

In Theorems 2.2–2.7, we proved interlacing properties for the zeros of

$${}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}; x\right)$$

and the zeros of polynomials contiguous to it, that is, those  ${}_2F_1$  polynomials where one of the parameters  $n, b$  or  $c$  is increased or decreased by unity. It is well known that the zeros of a polynomial depend continuously on its coefficients so it is a very natural question to ask what happens with respect to interlacing when we consider the zeros of a polynomial with parameter values in between those of the two contiguous polynomials. More specifically, can we prove an interlacing property for the zeros of (say)

$${}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}; x\right), \quad {}_2F_1\left(\begin{matrix} -n, b+t \\ c \end{matrix}; x\right), \quad 0 < t < 1,$$

and

$${}_2F_1\left(\begin{matrix} -n, b+1 \\ c \end{matrix}; x\right)?$$

It turns out that the interlacing situation is indeed what we might expect and the same method of proof used in Section 2 is applicable. We shall restrict ourselves, to a statement and proof of a generalisation of Theorem 2.2, the corresponding results for the other Theorems in Section 2 can be proved in a similar way.

**Theorem 3.1.** *Let  $c > 0$  and  $b+t > c+n-1$ ,  $0 < t < 1$  and let*

$$F_n(x) = {}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}; x\right).$$

*Then, the zeros*

$$0 < x_1 < x_2 < \cdots < x_n < 1 \text{ of } F_n(x),$$

$$0 < q_1 < q_2 < \cdots < q_n < 1 \text{ of } {}_2F_1\left(\begin{matrix} -n, b+t \\ c \end{matrix}; x\right)$$

*and*

$$0 < X_1 < X_2 < \cdots < X_n < 1 \text{ of } F_n(b+; x)$$

*are interlaced as follows:*

$$0 < X_1 < q_1 < x_1 < X_2 < q_2 < x_2 < \cdots < X_n < q_n < x_n < 1. \quad (8)$$

**Proof.** For  $c > 0$  and  $b+t > c+n-1$  the polynomial  $F_n(x)$  is orthogonal on  $(0, 1)$  with respect to the weight function  $w_1$  given by (3) whereas for any fixed  $t \in (0, 1)$  the weight function associated with the polynomial

$${}_2F_1\left(\begin{matrix} -n, b+t \\ c \end{matrix}; x\right),$$

on the same interval is given by  $w_6 = x^{c-1}(1-x)^{b-c-n+t}$ . The ratio of the weight functions is then

$$\frac{w_1(x)}{w_6(x)} = \frac{1}{(1-x)^t}$$

which is an increasing function of  $x \in (0, 1)$ . We deduce from Markoff's Theorem that

$$q_k < x_k \quad \text{for each } k = 1, 2, \dots, n. \quad (9)$$

The weight function  $w_2$  associated with  $F_n(b+; x)$  on  $(0, 1)$  is given by (4) and the ratio of the weight functions

$$\frac{w_6(x)}{w_2(x)} = (1-x)^{t-1}$$

is an increasing function of  $x \in (0, 1)$ . Therefore, we have

$$X_k < q_k \quad \text{for each } k = 1, 2, \dots, n. \quad (10)$$

We know from Theorem 2.2 that  $x_k < X_{k+1}$  for  $k = 1, 2, \dots, n-1$ , which, together with (9) and (10) means that  $0 < X_1 < q_1 < x_1 < X_2 < q_2 < x_2 < \dots < X_n < q_n < x_n < 1$  and the proof of (8) is complete.  $\square$

**Remark.** The interlacing properties of the zeros of contiguous hypergeometric polynomials are equivalent to interlacing properties for the zeros of Jacobi polynomials  $\mathcal{P}_n^{(\alpha, \beta)}(x)$  with the parameters  $\alpha$  and  $\beta$  increased or decreased by unity. In place of the contiguous function relations, one uses the corresponding identities for Jacobi polynomials, see for example [6, Chapter 16, pp. 263–265].

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